Invariant Polynomial Functions on k qudits

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February 1, 2008

Abstract

We study the polynomial functions on tensor states in $(\mathbb{C}^n)^{\otimes k}$ which are invariant under $SU(n)^k$. We describe the space of invariant polynomials in terms of symmetric group representations. For k even, the smallest degree for invariant polynomials is n and in degree n we find a natural generalization of the determinant. For n, d fixed, we describe the asymptotic behavior of the dimension of the space of invariants as $k \to \infty$. We study in detail the space of homogeneous degree 4 invariant polynomial functions on $(\mathbb{C}^2)^{\otimes k}$.

1 Introduction

In quantum mechanics, a combination of states in Hilbert spaces H_1, .., H_k leads to a state in the tensor product Hilbert space $H_1 \otimes \cdots \otimes H_k$. Such a state will be called here a tensor state. In this paper we take $H_1 = \cdots = H_k = \mathbb{C}^n$ where $n > 1$. Then a tensor state is a joint state of k qudits. It would be very interesting to classify tensor states in $(\mathbb{C}^n)^{\otimes k}$ up to the action of the product $U(n)^k$ of unitary groups of local symmetries. A natural approach to this is to study the algebra of invariant polynomials. This approach was developed by Rains [R], by Grassl, Rötteler and Beth [G-R-B1] [G-R-B2], by Linden and Popescu [L-P] and by Coffman, Kundu and Wootters [C-K-W]. These authors study the ring of invariant polynomials in the components of a tensor state in $(\mathbb{C}^n)^{\otimes k}$ and in their complex-conjugates. For k qubits, explicit descriptions of invariants are given in [G-R-B1], [G-R-B2], [L-P] and in [C-K-W].

In this paper the symmetry group we consider is the product $G = SU(n)^k$ of special unitary groups; one thinks of G as the special group of local symmetries. We study the G-invariant polynomial functions Q on the tensor states in $(\mathbb{C}^n)^{\otimes k}$ (we discuss in §2 how this is relevant to the description of the G-orbits). We consider polynomials in the entries of a tensor state, in other words, holomorphic polynomials.

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Let $\mathcal{R}_{n,k,d}$ be the space of homogeneous degree d polynomial functions on tensor states in $(\mathbb{C}^n)^{\otimes k}$. Let $\mathcal{R}^G_{n,k,d}$ be the space of G-invariants in $\mathcal{R}_{n,k,d}$. See §2 for more discussion. We reduce the problem of computing $\mathcal{R}^G_{n,k,d}$ to a problem in the invariant theory of the symmetric group $\mathfrak{S}_d$ (Proposition 2.1). In particular, $\mathcal{R}^G_{n,k,d}$ is non-zero only if d is a multiple of n. So the "first"case is $d = n$; we examine this in §3. We find that if k is odd then $\mathcal{R}^G_{n,k,n} = 0$ while if k is even then $\mathcal{R}^G_{n,k,n}$ is 1-dimensional. In the latter case we write down (§3) explicitly the corresponding invariant polynomial $P_{n,k}$ in $\mathcal{R}_{n,k,n}$; we find $P_{n,k}$ is a natural generalization of the determinant of a square matrix.

For fixed n, d the direct sum $\oplus_k \mathcal{R}_{n,k,d}$ is an associative algebra. We study the asymptotic behavior of $\dim \mathcal{R}^G_{n,k,d}$ as $k \to \infty$ in §4. In §5, we specialize to the case of k-qubits, i.e. $n = 2$. We compute the dimension of the space $\mathcal{R}^G_{2,k,4}$ of degree 4 invariants as well as the dimension of the space of invariants in $\mathcal{R}^G_{2,k,4}$ under the natural action of $\mathfrak{S}_k$. We show that $\oplus_k \mathcal{R}^{\mathfrak{S}_k \ltimes G}_{2,k,4}$ is a polynomial algebra on 2 generators. For $k \leq 5$ we describe the representation of $\mathfrak{S}_k$ on $\mathcal{R}^G_{2,k,4}$. For $k = 4$ we find some interesting relations with the results on classification of tensor states in $(\mathbb{C}^2)^{\otimes 4}$ given in [B].

We thank Markus Grassl for his useful comments on the first version of this paper.

2 Polynomial invariants of tensor states

We will consider $(\mathbb{C}^n)^{\otimes k}$ as a space of contravariant tensor states u. Then (once we fix a basis of $\mathbb{C}^n$) u is given by n^k components $u^{p_1 p_2 \cdots p_k}$. We consider the algebra $\mathcal{R}_{n,k}$ of polynomial functions on $(\mathbb{C}^n)^{\otimes k}$. So $\mathcal{R}_{n,k}$ is the polynomial algebra $\mathbb{C}[x_{p_1 p_2 \cdots p_k}]$ in the n^k coordinate functions $x_{p_1 p_2 \cdots p_k}$. We have a natural algebra grading $\mathcal{R}_{n,k} = \oplus_{d=0}^{\infty} \mathcal{R}_{n,k,d}$ where $\mathcal{R}_{n,k,d}$ is the space of homogeneous degree d polynomial functions.

A function in $\mathcal{R}_{n,k,d}$ amounts to a symmetric degree d covariant tensor Q in $(\mathbb{C}^n)^{\otimes k}$. So Q has n^{dk} components $Q_{i_{11} \cdots i_{dk}}$ where we think of the indices i_{ab} as being arranged in a rectangular array of d rows and k columns and $Q_{i_{11} \cdots i_{dk}}$ is invariant under permutations of the rows of the array. Then Q defines the function

$$u \mapsto Q_{i_{11} \cdots i_{dk}} u^{i_{11} i_{12} \cdots i_{1k}} u^{i_{21} i_{22} \cdots i_{2k}} \cdots u^{i_{d1} i_{d2} \cdots i_{dk}} \tag{2.1}$$

where we used the usual Einstein summation convention. In this way, $\mathcal{R}_{n,k}$ identifies with $S^d((\mathbb{C}^n)^{\otimes k})$.

Now the group $G = SU(n)^k$ acts on our tensor states u and tensors Q as follows. Let the matrix g_{ij} live in the m-th copy of $SU(n)$ and let g^{ij} be the inverse matrix. Then g_{ij} transforms $u^{p_1 p_2 \cdots p_k}$ into $g_{p_m q_m} u^{q_1 q_2 \cdots \cdots q_k}$ and $Q_{i_{11} \cdots i_{dk}}$ into $Q_{j_{11} \cdots j_{dk}} g^{j_{1m} i_{1m}} g^{j_{2m} i_{2m}} \cdots g^{j_{dm} i_{dm}}$. The identification of $\mathcal{R}_{n,k,d}$ with $S^d((\mathbb{C}^n)^{\otimes k})$ is G-equivariant.

We are interested in the algebra $\mathcal{R}^G_{n,k} = \oplus_{d=0}^{\infty} \mathcal{R}^G_{n,k,d}$ of G-invariants. We view this as a first step towards studying the orbits of G on $(\mathbb{C}^n)^{\otimes k}$. One can first study the orbits of the complex group $G_{\mathbb{C}} = SL(n, \mathbb{C})^k$ and then decompose the $G_{\mathbb{C}}$-orbits under the G-action. Note that a polynomial is G-invariant if and only if its is $G_{\mathbb{C}}$-invariant. The closed $G_{\mathbb{C}}$ orbits play a special role – they are the most degenerate orbits. Given any orbit Y, its closure contains a unique closed orbit Z; then points in Y degenerate to

points in Z. The $G_{\mathbb{C}}$-invariant functions separate the closed orbits; they take the same values on Y and on Z. The set of closed orbits of $G_{\mathbb{C}}$ in $(\mathbb{C}^n)^{\otimes k}$ has the structure of an affine complex algebraic variety with $\mathcal{R}^G_{n,k}$ as its algebra of regular functions. Thus a complete description of $\mathcal{R}^G_{n,k}$ would lead to a precise knowledge of the closed $G_{\mathbb{C}}$-orbits.

Our approach is thus somewhat different from that of [R] [G-R-B1] [G-R-B2] [L-P] [C-K-W] who study the invariant functions on $(\mathbb{C}^n)^{\otimes k}$ which are polynomials in the $x_{p_1\cdots p_k}$ and in their complex conjugates; these can also be described as the invariant polynomial functions on $(\mathbb{C}^n)^{\otimes k} \oplus \overline{(\mathbb{C}^n)^{\otimes k}}$.

At this point it is useful to examine the case $k = 2$. We can identify $(\mathbb{C}^n)^{\otimes 2}$ with the space $M_n(\mathbb{C})$ of square matrices and then $G = SU(n)^2$ acts on $M_n(\mathbb{C})$ by $(g,h)\cdot u = guh^{-1}$. So $\mathcal{R}^G_{n,k,d}$ is the space of homogeneous degree d polynomial functions Q of an n by n matrix u which are bi-$SL(n,\mathbb{C})$-invariant, i.e. $Q(guh^{-1}) = Q(u)$ for $g,h \in SL(n,\mathbb{C})$. Then Q is, up to scaling, the rth power of the determinant D for some r. Hence $d = rn$. It follows that $\mathcal{R}^G_{n,2}$ is the polynomial algebra $\mathbb{C}[D]$. Thus the space of closed orbits for $SL(n,\mathbb{C})^2$ identifies with $\mathbb{C}$, where λ corresponds to the unique closed orbit Z_λ inside the set X_λ of matrices of determinant λ. For $\lambda \neq 0$, $Z_\lambda = X_\lambda$ while for $\lambda = 0$, Z_0 reduces to the zero matrix.

We view $S^d((\mathbb{C}^n)^{\otimes k})$ as the space of invariants for the symmetric group $\mathfrak{S}_d$ acting on $((\mathbb{C}^n)^{\otimes k})^{\otimes d}$. So

$$\mathcal{R}^G_{n,k,d} = (((\mathbb{C}^n)^{\otimes k})^{\otimes d})^{G\times\mathfrak{S}_d} = (((\mathbb{C}^n)^{\otimes d})^{\otimes k})^{G\times\mathfrak{S}_d} \tag{2.2}$$

Recall the Schur decomposition $(\mathbb{C}^n)^{\otimes d} = \oplus_\alpha\, S^\alpha(\mathbb{C}^n)\otimes E_\alpha$ where α ranges over partitions of d with at most n rows, $S^\alpha(\mathbb{C}^n)$ is the irreducible covariant representation of $SU(n)$ given by the Schur functor S^α, and E_α is the corresponding irreducible representation of $\mathfrak{S}_d$. We use the convention that E_α is the trivial representation if $\alpha = [d]$, while E_α is the sign representation if $\alpha = [1^d]$. Thus we have

$$((\mathbb{C}^n)^{\otimes d})^{\otimes k} = \sum_{|\alpha_1|=\cdots=|\alpha_k|=d} S^{\alpha_1}(\mathbb{C}^n)\otimes\cdots\otimes S^{\alpha_k}(\mathbb{C}^n)\otimes E_{\alpha_1}\otimes\cdots\otimes E_{\alpha_k} \tag{2.3}$$

Now taking the invariants under $G\times\mathfrak{S}_d$ we get

$$\mathcal{R}^G_{n,k,d} = \sum_{|\alpha_1|=\cdots=|\alpha_k|=d} S^{\alpha_1}(\mathbb{C}^n)^{SU(n)}\otimes\cdots\otimes S^{\alpha_k}(\mathbb{C}^n)^{SU(n)}\otimes (E_{\alpha_1}\otimes\cdots\otimes E_{\alpha_k})^{\mathfrak{S}_d} \tag{2.4}$$

The representation $S^{\alpha_j}(\mathbb{C}^n)$, since it is irreducible, has no $SU(n)$-invariants except if $S^{\alpha_j}(\mathbb{C}^n) = \mathbb{C}$ is trivial. This happens if and only if α_j is a rectangular partition with all columns of length n. This proves:

Proposition 2.1. *If n does not divide d, then $\mathcal{R}^G_{n,k,d} = 0$. If $d = nr$, then $\mathcal{R}^G_{n,k,d}$ is isomorphic to $(E_\pi^{\otimes k})^{\mathfrak{S}_d}$ where $\pi = [r^n]$.*

The permutation action of $\mathfrak{S}_k$ on $(\mathbb{C}^n)^{\otimes k}$ induces an action of $\mathfrak{S}_k$ on $\mathcal{R}^G_{n,k,d}$.

Corollary 2.2. *The isomorphism of Proposition 2.1 intertwines the $\mathfrak{S}_k$-action on $\mathcal{R}^G_{n,k,d}$ with the action of $\mathfrak{S}_k$ on $(E_\pi^{\otimes k})^{\mathfrak{S}_d}$ given by permuting the k factors E_π.*

3 The generalized determinant function

Given n and k, we want to find the smallest positive value of d such that $\mathcal{R}^G_{n,k,d} \neq 0$. By Proposition 2.1, the first candidate is $d = n$.

Corollary 3.1. *$\mathcal{R}^G_{n,k,n} \neq 0$ iff k is even. In that case, $\mathcal{R}^G_{n,k,n}$ is one-dimensional and consists of the multiples of the function $P_{n,k}$ given by*

$$P_{n,k}(u) = \sum_{\sigma_2,\cdots,\sigma_k\in\mathfrak{S}_n} \epsilon(\sigma_2)\cdots\epsilon(\sigma_k) \ \prod_{h=1}^n \ u^{hh_{\sigma_2}\cdots h_{\sigma_k}} \tag{3.1}$$

where $h_{\sigma_j} = \sigma_j(h)$.

Proof. By Proposition 2.1, we need to compute $(E_\pi^{\otimes k})^{\mathfrak{S}_d}$. For $d = n$, $\pi = [1^n]$ and so E_π is the sign representation of $\mathfrak{S}_n$. Then $(E_\pi^{\otimes k})$ is one-dimensional and carries the trivial representation if k is even, or the sign representation if k is odd.

Now for k even, we can easily compute a non-zero function $P = P_{n,k}$ in $\mathcal{R}_{n,k,n}$. For $S^\pi(\mathbb{C}^n)$ is the top exterior power $\wedge^n\mathbb{C}^n$. Thus P is a non-zero element of the one-dimensional subspace $(\wedge^n\mathbb{C}^n)^{\otimes k}$ of $((\mathbb{C}^n)^{\otimes n})^{\otimes k}$. The tensor components of P are then given by $P_{i_{11}\cdots i_{nk}} = \frac{1}{n!}\epsilon(\sigma_1)\cdots\epsilon(\sigma_k)$ if for each j, the column $i_{1j},\cdots,i_{nj}$ is a permutation σ_j of $1,\cdots,n$ and 0 otherwise. Then we get

$$P_{n,k}(u) = \frac{1}{n!}\sum_{\sigma_1,\cdots,\sigma_k\in\mathfrak{S}_n} \epsilon(\sigma_1)\cdots\epsilon(\sigma_k) \ \prod_{h=1}^n \ u^{h_{\sigma_1}\cdots h_{\sigma_k}} \tag{3.2}$$

where $h_{\sigma_i} = \sigma_i(h)$. The expression is very redundant, as each term appears $n!$ times. We remedy this by restricting the first permutation σ_1 to be 1. This gives (3.1). $\square$

$P_{n,k}$ is a *generalized determinant*; $P_{n,k}$ is invariant under the $\mathfrak{S}_k$-action. For $k = 2$, (3.1) reduces to the usual formula for the matrix determinant.

Recall that the rank s of a tensor state u in $(\mathbb{C}^n)^{\otimes k}$ is the smallest integer s such that u can be written as $u = v_1 + v_2 + \cdots + v_s$, where the v_i are decomposable tensor states $v_i = w_{i1}\otimes w_{i2}\otimes\cdots\otimes w_{ik}$. There is a relation between the rank and the vanishing of $P_{n,k}$ as follows:

Corollary 3.2. *If the tensor state u in $(\mathbb{C}^n)^{\otimes k}$ has rank less than n, then $P_{n,k}(u) = 0$.*

It is easy to find a tensor state u of rank n such that $P_{n,k}(u)$ is non-zero. For instance, $P_{n,k}(u) = 1$ if u has all components zero except $u^{1\cdots1} = \cdots = u^{n\cdots n} = 1$. For $k = 2$, $P_{n,k}(u) = 0$ implies u has rank less than n. For bigger (even) k, this is false, if n is large enough. This happens essentially because the rank of u can be very large (at least $\frac{n^k}{kn-k+1}$). Thus $P_{n,k}$ gives only partial information about the rank.

4 Asymptotics as $k \to \infty$

Suppose we fix n and d where $d = rn$. Then there is a G-invariant associative graded algebra structure $P \circ Q$ on the direct sum $\oplus_k\mathcal{R}^G_{n,k,d}$. Indeed, the product of tensors

induces a $(G\times\mathfrak{S}_d)$-invariant map $V^{\otimes k}\otimes V^{\otimes l}\to V^{\otimes(k+l)}$ where $V=(\mathbb{C}^n)^{\otimes d}$. The induced multiplication on the spaces of $(G\times\mathfrak{S}_d)$-invariants gives the product on $\oplus_k\mathcal{R}^G_{n,k,d}$, where we use the identification in (2.2). This multiplication corresponds, under the isomorphism of Proposition 2.1, to the product map $E_\pi^{\otimes k}\otimes E_\pi^{\otimes l}\to E_\pi^{\otimes(k+l)}$. This structure is very useful. For instance, if $d=n$, then $P_{n,k}\circ P_{n,l}=\frac{1}{n!}P_{n,k+l}$. Thus the determinant $P_{n,2}$ determines $P_{n,2m}$ in that the m-fold product $P_{n,2}\circ\cdots\circ P_{n,2}$ is equal to $(n!)^{-m+1}P_{n,2m}$.

We will study the size of the algebra $\oplus_k\mathcal{R}^G_{n,k,d}$ by finding an asymptotic formula for the dimension of $\mathcal{R}^G_{n,k,d}$. We do this for $r\geq 2$. Indeed for $r=1$ we already know $\dim\mathcal{R}^G_{n,k,n}$ is 1 if k is even or 0 if k is odd; we call this the *static* case. The asymptotics involve the number

$$p=\dim E_\pi=d!\prod_{m=0}^{n-1}\frac{m!}{(m+r)!} \tag{4.1}$$

where $\pi=[r^n]$ as in Proposition 2.1. Our formula for p is immediate from the hook formula for the dimension of an irreducible symmetric group representation.

Proposition 4.1. *Assume $d=rn$ with $r\geq 2$. Then* $\dim\mathcal{R}^G_{n,k,d}\sim c\dfrac{p^k}{d!}$ *as $k\to\infty$, where $c=1$ with one exception: $c=4$ if $n=2,d=4$.*

Proof. Let $s=\dim\mathcal{R}^G_{n,k,d}=\dim(E_\pi^{\otimes k})^{\mathfrak{S}_d}$. Then $s=\frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d}\chi(\sigma)^k$ where $\chi:\mathfrak{S}_d\to\mathbb{Z}$ is the character of E_π. If σ acts trivially on E_π, then $\chi(\sigma)=p$. If σ acts non-trivially, we claim $|\chi(\sigma)|<p$. To show this, it suffices to show that σ has at least two distinct eigenvalues on E_π; this is because $\chi(\sigma)$ is the sum of the p eigenvalues of σ. Now the set $\mathfrak{T}_d$ of $\sigma\in\mathfrak{S}_d$ which act on E_π by a scalar is a normal subgroup of $\mathfrak{S}_d$. So if $d\geq 5$, then $\mathfrak{T}_d$ is $\{1\}$, the alternating group $\mathfrak{A}_d$ or $\mathfrak{S}_d$. We can easily rule out the latter two possibilities, so $\mathfrak{T}_d=\{1\}$, which proves our claim. If $d\leq 4$, then (since $r>1$ and $n>1$), we have $d=4$, $n=2$ and $\pi=[2,2]$. Our claim is clear here since $\mathfrak{S}_4$ acts on E_π through the reflection representation of $\mathfrak{S}_3$ on $\mathbb{C}^2$.

Therefore we have $s=c\frac{p^k}{d!}+o(p^k)$ as $k\to\infty$ where c is cardinality of the kernel of $\mathfrak{S}_d\to Aut\,E_\pi$. Our work in the previous paragraph computes c. ∎

Proposition 4.1 implies that the algebra $\oplus_k\mathcal{R}^G_{n,k,d}$ is far from commutative, as it has roughly $1/N$ times the size of the tensor algebra $\oplus_k(\mathbb{C}^p)^{\otimes k}$. We note however that the $\mathfrak{S}_k$-invariants in $\oplus_k\mathcal{R}^G_{n,k,d}$ form a commutative subalgebra, isomorphic to $S(E_\pi)^{\mathfrak{S}_d}$.

5 Quartic invariants of k qubits

The case $n=2$ is of particular interest, as here the qudits are qubits, and this is the case being most discussed in quantum computation. Here we can give some precise non-asymptotic results for the first non-static case, namely $\mathcal{R}^G_{2,k,4}$. We put $E=E_\pi=E_{[2,2]}$. The proof of Proposition 4.1 easily gives

Corollary 5.1. *We have* $\dim\mathcal{R}^G_{2,k,4}=\frac{1}{3}(2^{k-1}+(-1)^k)$.

The first few values of $\dim \mathcal{R}^G_{2,k,4}$, starting at $k = 1$, are $0, 1, 1, 3, 5, 11, 21, 43$. For $k = 2$ and $k = 3$ the unique (up to scalar) invariants are, respectively, the squared determinant $P^2_{2,2}$ and the Cayley hyperdeterminant $H_{2,3}$ (see [G-K-Z]). We note that the hyperdeterminant is very closely related to the relative tangle of 3 entangled qubits discussd in [C-K-W].

It would be useful to study $\mathcal{R}^G_{2,k,4}$ as a representation of $\mathfrak{S}_k$, where $\mathfrak{S}_k$ acts by permuting the k qubits. The $\mathfrak{S}_k$-invariants in $\mathcal{R}^G_{2,k,4}$ are the $(\mathfrak{S}_k \ltimes G)$-invariants in $\mathcal{R}_{2,k,4}$. These $(\mathfrak{S}_k \ltimes G)$-invariant polynomials are very significant as they separate the closed orbits of the extended symmetry group $\mathfrak{S}_k \ltimes SL(2, \mathbb{C})^k$ acting on $(\mathbb{C}^2)^{\otimes k}$. We can compute the dimension of the $\mathfrak{S}_k$-invariants as follows:

Proposition 5.2. *The dimension of the space of $\mathfrak{S}_k \ltimes G$-invariants in $\mathcal{R}_{2,k,4}$ is $M_k = \left[\frac{k}{6}\right] + r_k$ where $r_k = 0$ if $k \equiv 1 \bmod 6$, or $r_k = 1$ otherwise. Furthermore the algebra $\oplus_k \mathcal{R}^{\mathfrak{S}_k \ltimes G}_{2,k,4}$ is the polynomial algebra $\mathbb{C}[P^2_{2,2}, H_{2,3}]$.*

Proof. We have isomorphisms $\mathcal{R}^{\mathfrak{S}_k \ltimes G}_{2,k,4} \simeq (E^{\otimes k})^{\mathfrak{S}_k \times \mathfrak{S}_3} \simeq S^k(E)^{\mathfrak{S}_3}$ since the representation of $\mathfrak{S}_4$ on E factors through $\mathfrak{S}_3$. Thus the algebra $\oplus_k \mathcal{R}^{\mathfrak{S}_k \ltimes G}_{2,k,4}$ identifies with $S(E)^{\mathfrak{S}_3}$. Now $S(E)^{\mathfrak{S}_3}$ is the algebra of $\mathfrak{S}_3$-invariant polynomial functions on traceless 3×3 diagonal matrices, and so is a polynomial algebra on the functions $A \mapsto Tr(A^2)$ and $A \mapsto Tr(A^3)$. These invariants correspond (up to scaling) to $P^2_{2,2}$ and $H_{2,3}$. The formula for the dimension follows easily. $\square$

For instance, we have: $M_1 = 0$, $M_k = 1$ for $2 \leq k \leq 5$, and $M_6 = 2$. We remark that by replacing $S(E)^{\mathfrak{S}_3}$ by $\wedge(E)^{\mathfrak{S}_3}$, it is easy to prove that the sign representation of $\mathfrak{S}_k$ does not occur in $(E^{\otimes k})^{\mathfrak{S}_4}$ for any $k \geq 2$.

We can determine the $\mathfrak{S}_k$-representation on $\mathcal{R}^G_{2,k,4}$ for small k by explicit trace computations. For $k = 2$ and $k = 3$ we have the trivial 1-dimensional representation. For $k = 4$, we find $\mathcal{R}^G_{2,4,4}$ is the direct sum $E_{[4]} \oplus E_{[2,2]}$. The trivial representation $E_{[4]}$ of $\mathfrak{S}_4$ is spanned by $P^2_{2,4}$, while the 2-dimensional representation $E = E_{[2,2]}$ is spanned by the determinants $\Delta(ijkl)$ introduced in [B]. Here $(ijkl)$ is a permutation of (1234). Given a tensor state $u \in (\mathbb{C}^2)^{\otimes 4}$, we can view it as an element v of $\mathbb{C}^4 \otimes \mathbb{C}^4$, where the first (resp. second) $\mathbb{C}^4$ is the tensor product of the i-th and j-th copies of $\mathbb{C}^2$ (resp. of the k-th and l-th copies). Then $\Delta(ijkl)(u)$ is the determinant of v. As shown in [B], the $\Delta(ijkl)$ span the representation E of $\mathfrak{S}_4$. The significance of the $\Delta(ijkl)$ is that their vanishing describes the closure of the set of tensor states in $(\mathbb{C}^2)^{\otimes 4}$ of rank ≤ 3. For $k = 5$ the representation $\mathcal{R}^G_{2,5,4}$ of $\mathfrak{S}_5$ is $E_{[5]} \oplus E_{[2,1,1,1]}$.

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